Homework 5 Sample Solutions

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4. Roll a fair die twice. Let X be the random variable that gives the maximum of the two numbers. Find the probability mass function describing the distribution of X.

   Solution: Observe that our sample space Ω ($|\Omega| = 36$) can be written as follows:

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

   From this, we should observe that 1 will occur as the maximum 1 time, 2 will occur as the maximum 3 times, 3 will occur as the maximum 5 times, 4 will occur as the maximum 7 times, 5 will occur as the maximum 9 times, and 6 will occur as the maximum 11 times (if you are having trouble seeing this, look at the rows and columns of the way the sample space is drawn above). Thus, we can write our probability mass function as:

$$\begin{aligned}P(X=1) &= \tfrac{1}{36}\\ P(X=2) &= \tfrac{3}{36}\\ P(X=3) &= \tfrac{5}{36}\\ P(X=4) &= \tfrac{7}{36}\\ P(X=5) &= \tfrac{9}{36}\\ P(X=6) &= \tfrac{11}{36}\end{aligned}$$

   As a sanity check, you can note that the sum of our probability mass function over all possible values for X is equal to 1.

   $\square$

6. An urn contains five green balls, two blue balls, and three red balls. You remove three balls at random without replacement. Let X denote the number of red balls. Find the probability mass function describing the distribution of X.

   Solution: Our random variable X can take on 4 possible values. From our removal of the three balls, we can end up with 0 red balls, 1 red ball, 2 red balls, or 3 red balls. Thus, we can have $X = 0$, $X = 1$, $X = 2$, or $X = 3$. Now, we should calculate the probability of each possible value of X. There are 10 balls in total, and we are picking 3 without replacement. Thus, there are $\binom{10}{3}$ total ways to remove three balls without replacement from the urn. There are 3 red balls and 7 non-red balls. The probabilities are calculated as follows:

   For $X = 0:$ We will have 0 red balls and 3 non-red balls. Thus, $P(X = 0) = \frac{\binom{3}{0}\binom{7}{3}}{\binom{10}{3}}$

   For $X = 1:$ We will have 1 red ball and 2 non-red balls. Thus $P(X = 1) = \frac{\binom{3}{1}\binom{7}{2}}{\binom{10}{3}}$

   For $X = 2:$ We will have 2 red balls and 1 non-red ball. Thus $P(X = 2) = \frac{\binom{3}{2}\binom{7}{1}}{\binom{10}{3}}$

   For $X = 3:$ We will have 3 red balls and 0 non-red balls. Thus $P(X = 3) = \frac{\binom{3}{3}\binom{7}{0}}{\binom{10}{3}}$

   We have our values of $P(X = x)$ for all possible X, and thus we have our probablity mass function describing the distribution of X.

   $\square$

8. You draw 5 cards from a standard deck of 52 cards without replacement. Let X denote the number of aces in your hand. Find the probability mass function describing the distribution of X.

   Solution: Our random variable X can take on 5 possible values. In our hand, we can either have 0 aces ($X = 0$), 1 ace ($X = 1$), 2 aces ($X = 2$), 3 aces ($X = 3$), or 4 aces ($X = 4$). *Note that we cannot have $X = 5$ as this would mean we have 5 aces in our hand, which is impossible to do given a standard deck of 52 cards.* We have $\binom{52}{5}$ ways of picking 5 cards from the deck of 52. There are 4 aces, and 48 non-aces in the deck. The probabilities are calculated as follows:

   For $X = 0:$ We will have 0 aces and 5 non-aces. Thus, $P(X = 0) = \frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}}$

   For $X = 1:$ We will have 1 ace and 4 non-aces. Thus, $P(X = 1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$

   For $X = 2:$ We will have 2 aces and 3 non-aces. Thus, $P(X = 2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$

   For $X = 3:$ We will have 3 aces and 2 non-aces. Thus, $P(X = 3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$

   For $X = 4:$ We will have 4 aces and 1 non-aces. Thus, $P(X = 4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$

   We have our values of $P(X = x)$ for all possible X, and thus we have our probablity mass function describing the distribution of X.

   $\square$

10. Suppose the probability mass function of a discrete random variable X is given by the following table:

x	$P(X = x)$
-1	0.2
-0.5	0.25
0.1	0.1
0.5	0.1
1	0.35

Find and graph the corresponding distribution function $F(x)$.

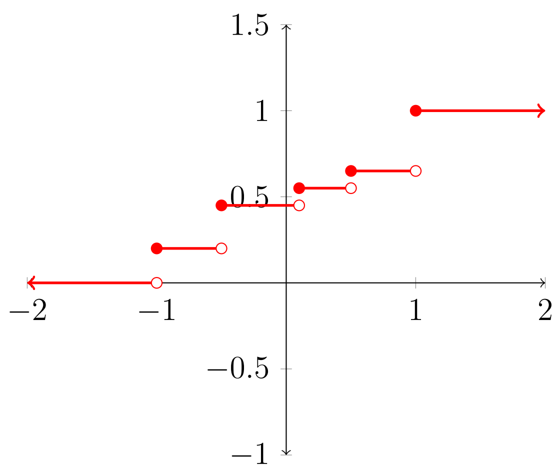
Solution: Recall that $F(x) = P(X \leq x)$. $F(x)$ is defined for all real numbers, and we must pay special attention to the cases when $X = x$ while building our function $F(x)$. We should observe that

$F(-1) = 0.2$,
$F(-0.5) = 0.2 + 0.25 = 0.45$,
$F(0.1) = 0.2 + 0.25 + 0.1 = 0.55$,
$F(0.5) = 0.2 + 0.25 + 0.1 + 0.1 = 0.65$, and
$F(1) = 0.2 + 0.25 + 0.1 + 0.1 + 0.35 = 1$.

Thus, $F(x)$ is a piecewise function as follows:

$$F(x) = \begin{cases} 0 & & x < -1 \\ 0.2 & -1 \leq & x < -0.5 \\ 0.45 & -0.5 \leq & x < 0.1 \\ 0.55 & 0.1 \leq & x < 0.5 \\ 0.65 & 0.5 \leq & x < 1 \\ 1 & 1 \leq & x \end{cases}$$

And our graph for $F(x)$ is drawn as follows:



□

12. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.05 & 0 \leq x < 1.3 \\ 0.30 & 1.3 \leq x < 1.7 \\ 0.85 & 1.7 \leq x < 1.9 \\ 0.90 & 1.9 \leq x < 2 \\ 1.0 & 2 \leq x \end{cases}$$

Determine the probability mass function of X.

Solution: We should look at the points of $F(x)$ "jumps". This will give us the points where $P(X = x)$ has a nonzero value. The size of the "jump" will be equal to the probability that X takes on at that value. By looking at our distribution function $F(x)$, we should observe that these "jumps" occur at $X = 0$, $X = 1.3$, $X = 1.7$, $X = 1.9$, and $X = 2$. Furthermore, by looking at the corresponding size of these "jumps" we obtain our probability mass function:

$$P(X = 0) = 0.05$$
$$P(X = 1.3) = 0.25$$
$$P(X = 1.7) = 0.55$$
$$P(X = 1.9) = 0.05$$
$$P(X = 2) = 0.10$$

□